System Identification (CH 5230)

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Models can assume various forms depending on the complexity of the physical phenomena that is being modelled and what the user assumes about the phenomena.

In empirical modelling, the biggest advantage is the flexibility of the model structure and complexity.

- The structure and complexity are largely driven by the end-use of the model rather than by process complexities (alone)
- For e.g., for a given process, a model for control is required to be simple and not as accurate as in soft-sensing where the model is expected to be more accurate and can possess a complex structure.

In order to choose a model structure, or in general, a candidate model, one needs to be aware of the various families of models that exist.

Models are classified according to what they capture in a process.

- The user chooses a candidate model based on (i) which phenomenon is being explained (ii) ease of estimation and (iii) end-use requirements of model.

The idea is always to explain process behaviour using inputs and observed (measured) data.

Classification of Models

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Models for identification

In identification, models are built from experimental data. Such data is obtained by exciting the process with an input and observing its response at regular intervals (discrete-time data).

- Empirical models for dynamic processes, therefore, are usually in discrete-time domain.
- Among the numerous possibilities of models, we select a model that (i) suitsly describes the process (ii) is easy to estimate and (iii) is convenient to implement.
- The simplest one is to begin with a lumped parameter, linear and time-invariant description.
  - Observe the usage of the term description - we are only attempting to describe process behaviour.
- How justified is the LTI description for a process?
Deterministic vs. Stochastic Models

- **A deterministic model** predicts the process behaviour accurately.
  - Deterministic models are used to explain processes whose physics is completely understood and usually associated with an external cause.

- **A stochastic model** predicts a process with less than 100% confidence (not accurate) - time-series models
  - Stochastic models are used to explain the response of a process for which no external cause can be associated OR the cause itself is not measured or known.

No process is completely predictable. Why?

- (i) The physics is never completely understood
- (ii) Random effects (noise) always exist

The response of any (physical) process may be split into two components:

- associated to an external cause (designed or known) - deterministic model
- associated to disturbances, noise, unmeasured inputs, modelling errors - stochastic model

Overall Model

- It is typically assumed that the stochastic component of the observed response is an additive one, i.e., adds on to the deterministic part
- Secondly, it is possible to assume (under certain conditions) that the stochastic part of the response is due to a fictitious random input (shock wave)

Linear Time-Invariant Systems

The assumption is that over the range of operation, the deterministic and stochastic components of the processes are time-invariant (stationary) and exhibit linear behaviour

**What is a linear system?**

A system is said to be linear if and only if it satisfies the (i) principle of homogeneity (scaled input should result in scaled output) and (ii) principle of superposition (a linear combination of inputs should produce the same linear combination of respective outputs)

- Eg., \( y = au + b \) is linear only if \( b = 0 \); however, \( (y - y_0) = a(u - u_0) \) is linear

In general, all processes are non-linear. Then how is a linear model justified?

- The linear model definitely carries an approximation error. It gives reasonably good predictions of the process behaviour for "small" changes around an operating point.
- There are several processes for which the linear models serve as good approximations. However, linear models have to be re-built for changes in operating conditions (multiple linear models)

Linear models are easier to estimate. Non-linear models not only present difficulties in estimation but also demand the knowledge of the type of non-linearity in the process

- CAUTION: There is a large class of processes for which a linear description is grossly insufficient.

Time-Invariance

- **What is a time-invariant system?**
  - A system is said to be time-invariant if it produces the same output for the same input regardless of when the input was provided
  - Mathematically, if \( u(t) \) yields \( y(t) \), then \( u(t-T) \) should yield \( y(t-T), \forall T \)
  - Practically, it means the characteristics of the system do not change with time

Is any system truly time-invariant? No.

- However, over the period of experiment and operation, the changes are (assumed to be) negligible - i.e., they do not cause the output to change significantly

A time-invariant model is a good beginning point. The time-varying nature of a process can be accommodated by updating the model parameters on-line (adaptive estimation)

- TI models are a lot easier to handle from estimation perspective. TV models require too many parameters to estimate.
Descriptions for LTI systems

We focus on (discrete-time) LTI descriptions for any process

- The theory for LTI systems is a very matured one
- Once we understand estimation of LTI models, we can extend the principles to other complicated model forms

The behaviour of an LTI system can be described in several ways

- Convolution equation form
- Response form (impulse response / step response / frequency response)
- Input-Output difference equation
- Transfer function representation
- State-space representation

The choice of a particular form depends on a few factors

Notion of a True Model

- No model can truly and completely describe the response of a process.
  - There are always uncertainties in the form of disturbances, noise and modelling errors

- Then, is there anything like a true model? No.
  - However, there is a purpose to imagine that a true model exists and is of a certain form

The purpose is to assess the quality of an estimated model

Consider this situation. We have developed a method to estimate the parameters of a model. We would like to know how good this method is. Ideally, we would like the method to do well at least when the model structure is identical to some “true” structure.

- Therefore, the purpose of assuming a true model or a true parameter is to be able to say whether an estimator is able to deliver good estimate at least under certain minimal conditions.

Notation

- Discrete-time signals will be indicated by signals with functions of sampling instants in square brackets, e.g., $x[k]$ (likewise $x(t)$ for continuous-time signals)
- Inputs and outputs are denoted as $u[.]$ and $y[.]$ while states are denoted by $x[.]$ respectively
- The deterministic process and stochastic process are denoted by $G$ and $H$ respectively
- $T_s$ denotes the sampling interval
- $g[.]$ denotes the impulse response of the LTI system
- The forward shift operator is denoted by ‘$q$’ (backward operator by $q^{-1}$)
- The model parameter vector is usually denoted by $\theta$. 

Descriptions of Linear Time-Invariant Systems
Convolution equation

\[
g[k] = \sum_{n=-\infty}^{\infty} g[n]u[k-n] = \sum_{n=-\infty}^{\infty} g[k-n]u[n]
\]

Impulse response

- Once we know the impulse response of an LTI system, we can predict the system’s response to any other arbitrary input:
  - Surprising result? Not so surprising since the system is linear and time-invariant

- Any input signal \( u[k] \) can be written as a weighted sum of shifted impulses:
  \[
u[k] = \sum_{n=-\infty}^{\infty} u[n]\delta[k-n] \quad \Rightarrow \quad y[k] = \sum_{n=-\infty}^{\infty} u[n]\text{Response(}\delta[k-n]\text{)}
\]
  (linearity property)

- Denote the system’s response to an impulse \( \delta[k] \) by \( g[k] \) => \( \delta[k-n] \) produces \( g[k-n] \)
  (time-invariance property)

- Combining both results, we have:
  \[
y[k] = \sum_{n=-\infty}^{\infty} u[n]g[k-n] = \sum_{n=-\infty}^{\infty} g[n]u[k-n]
\]

Impulse response descriptions

- For an LTI system, knowing the impulse response amounts to knowing “everything” about the system’s characteristics
  - Given the impulse response, we can predict the response of the LTI system to any input
  - From the impulse response, we can easily comment on system’s characteristics such as causality, stability, etc.

- Generating impulse response for a discrete-time system is physically possible unlike the case of continuous-time systems
  - The d.t. impulse can be generated whereas the c.t. impulse is not physically realizable

- Impulse response descriptions of LTI systems are quite popular in empirical modelling of processes

FIR Models

- To calculate the response at any time \( k' \) with the convolution equation, we need to know impulse response coefficients, \( \{g[0], \ldots, g[k]\} \)
  - The size of coefficients increases as we look at large times
  - Therefore, the convolution form is known as infinite impulse response (IIR) model

- From identification viewpoint, it is not practical to estimate such models since we have to estimate infinite IR coefficients.

- Fortunately, for stable systems, the impulse response practically vanishes in a finite time, i.e., \( g[k] \approx 0 \) for \( k \geq M \) (a finite number)

- Thus, for stable systems, we can use a truncated form of convolution form
  \[
y[k] = \sum_{n=0}^{M} g[n]u[k-n]
\]
  - Such models are known as Finite Impulse Response (FIR) models

A system is causal if and only if \( g[k] = 0, k < 0 \). Further \( g[0] = 0 \) => strict causality

\( u[0] \) affects \( y[k] \) through \( g[k], u[1] \) through \( g[k-1], \ldots \), and \( u[k] \) through \( g[0] \).

Convolution and Difference equations, z-Transforms, Transfer functions

- In most situations, the cause precedes the effect, i.e., the output \( y[k] \) only depends on past and/or present inputs \( \Rightarrow \) causal system

- For causal systems, an impulse input should not produce any response before it is applied.

\( y[k] = \sum_{n=0}^{\infty} g[n]u[k-n] = \sum_{n=0}^{M} g[k-n]u[n] \)
Step response descriptions

- Many changes in processes are of “step” type. A unit step is defined as,
  \[ u[k] = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases} \]
  - Therefore, it is reasonable to characterize the LTI system in terms of step responses
- The step response can be derived from impulse response and vice versa
  \[ y_s[k] = \sum_{n=0}^{k} g[k-n] \]
  \[ g[k] = y_s[k] - y_s[k-1], \quad k \geq 0 \]
  - The typical impulse and response of a first-order system is shown above
  - The step response can be estimated by means of a step input (unlike the case of impulse response) or from the data generated by an arbitrary input.

Frequency response function (FRF)

- The frequency response function \( G(\omega) \) (sometimes written as \( G(e^{j\omega}) \)) is defined as the Fourier Transform of the impulse response function \( g[n] \)
- A few remarks, which have important implications:
  - A sine input to an LTI system produces a sine wave of the same frequency
  - The amount by which the amplitude is amplified/attenuated is given by the amplitude ratio, is characterized by the amplitude ratio of output to input, \( AR(\omega) = |G(\omega)| \) (Amplitude Ratio)
    - The amplitude ratio depends on the frequency of the input
  - The amount by which the sine wave is shifted, is a function of the frequency and given by \( \phi(\omega) = \angle G(\omega) \) (Phase Shift)
    - In essence, the LTI system treats each frequency differently - the FRF quantifies this information
- The knowledge of FRF gives us the “filtering” characteristics of the LTI system.

Frequency response

- The third form of description is in terms of its frequency response
  - Until now we have seen descriptions w.r.t. to two elementary signals: (i) impulse (ii) step
- The frequency response is defined as the response to sine/cosine inputs
  - Mathematically, the sine and cosine are special cases of the complex exponential, \( e^{j\omega t} \)

Using the convolution equation, one can write

\[ y[k] = \sum_{n=0}^{\infty} g[n]u[k-n] = \sum_{n=0}^{\infty} g[n]e^{j\omega(k-n)} = \sum_{n=0}^{\infty} g[n]e^{-j\omega n}e^{j\omega k} = \frac{G(\omega)e^{j\omega k}}{G(\omega)} \]

Thus, the output is also a sine (cosine) with the same frequency, but with modified amplitude and shifted phase.

- The quantity \( G(\omega) = \sum_{n=0}^{\infty} g[n]e^{-j\omega n} \), is known as the frequency response function.

Difference equation form

- Earlier, we developed the FIR model as an approximation of the IIR model
- We can, however, develop an accurate and alternative representation which requires much fewer parameters.

Example: An LTI system has impulse response, \( g[k] = b(-a)^k, \quad |a| < 1, \quad k \geq 0 \)

\[ y[k] = \sum_{n=0}^{k} g[k-n]u[k] = \sum_{n=0}^{k} b(-a)^{k-n}u[k] \]
\[ y[k-1] = \sum_{n=0}^{k-1} b(-a)^{k-1-n}u[k-1] = (-a)^{k-1} \sum_{n=0}^{k-1} b(-a)^{k-n}u[k-1] \]
\[ \Rightarrow y[k] + ay[k-1] = bu[k] \]

The final equation that we have is known as a difference equation, which is an accurate and alternative form of the convolution equation.

Observe that we have used past output as well to describe the present output.
**Difference equation descriptions**

- In general, any LTI system can be suitably described by a difference equation with constant coefficients:

\[
y[k] + a_1 y[k-1] + \cdots + a_n y[k-n_a] = b_0 u[k] + b_1 u[k-1] + \cdots + b_m u[k-n_b]
\]

Observe the difference between the convolution form and the above form:

- The output is expressed as a weighted sum of finite number of past inputs and outputs.

- The convolution form and difference forms are interconvertible:
  - If the system has an IIR representation, then it has a difference equation description with finite parameters.
  - An FIR model is, in fact, a special case of difference equation form with only past inputs.

- The number of past outputs that influence the present output is said to be the order of the LTI system.
  - E.g.: \(y[k] = -0.5 y[k-1] + 2 u[k-2]\) (first-order); \(y[k] = -1.2 y[k-2] + y[k-3] + 1.5 y[k-1]\) (second-order).

**Shift-operator forms**

- The difference equation forms (as well as IIR forms) can be conveniently represented in terms of a shift-operator.

- A backward shift operator \(q^{-1}\) is defined such that when it operates on a sample, it produces the previous sample:

\[
q^{-1} x[k] = x[k-1] \Rightarrow q^{-n} x[k] = x[k-n]
\]

- A forward shift operator is defined in a similar way: \(q x[k] = x[k+1]\).

- Note that it is an operator and not a multiplier!

- The difference equation form can be written in terms of this operator:

\[
(1 + a_1 q^{-1} + \cdots + a_n q^{-n_a}) y[k] = (b_0 + b_1 q^{-1} + \cdots + b_m q^{-n_b}) u[k]
\]

\[
y[k] = \frac{b_0 + b_1 q^{-1} + \cdots + b_m q^{-n_b}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n_a}} u[k]
\]

OR

\[
y[k] = G(q^{-1}) u[k]
\]

**Difference vs. Convolution form**

<table>
<thead>
<tr>
<th>Difference form</th>
<th>Convolution form</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output is expressed as a weighted sum of past outputs and inputs</td>
<td>Output is expressed as a weighted sum of past inputs only</td>
</tr>
<tr>
<td>The model has a specific structure - i.e., the no. of past outputs and past inputs</td>
<td>No assumption on the structure is required - it is a fundamental equation for LTI systems</td>
</tr>
<tr>
<td>The parameters do not directly reflect the system’s characteristics</td>
<td>The coefficients have a direct connection with the system’s characteristics</td>
</tr>
<tr>
<td>Model is characterized by usually a few parameters</td>
<td>Model usually requires a larger set of coefficients</td>
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</tbody>
</table>

- A prime distinction is the requirement of the knowledge of the structure in the difference equation form:
  - The convolution form is fundamental to LTI systems. No other assumption besides LTI is required.

- The difference forms are, on the other hand, very powerful because of the parsimony in parameters and their suitability for computation.

**Transfer function operator**

- The quantity \(G(q^{-1})\) is known as the transfer function operator.

\[
G(q^{-1}) = \frac{b_0 + b_1 q^{-1} + \cdots + b_m q^{-n_b}}{1 + a_1 q^{-1} + \cdots + a_n q^{-n_a}}
\]

- It denotes how much of the input is transferred to the output.

- It is NOT a multiplier but is in fact an operator \(\Rightarrow\) hence the name.

- The transfer function operator is a ratio of two polynomials (in operator).

- The operator can be written in terms of forward shift operator as well.

**Example:** \(y[k] = 1.2 y[k-1] + 0.32 y[k-2] = u[k-1] + 0.8 u[k-2]\)

\[
G(q^{-1}) = \frac{1 + 0.8 q^{-1}}{1 - 1.2 q^{-1} + 0.32 q^{-2}}
\]

OR

\[
G(q) = \frac{q^2 + 0.8 q}{q^2 - 1.2 q + 0.32}
\]
Difference to Convolution forms

**Example:** \( y[k] - 0.5y[k-1] = u[k-1] \) \( \Rightarrow \) \( G(q^{-1}) = \frac{q^{-1}}{1 - 0.5q^{-1}} \)

\[
y[k] = (1 - 0.5q^{-1})^{-1} q^{-1} u[k] = (1 + 0.5q^{-1} + 0.25q^{-2} + 0.125q^{-3} + \cdots) q^{-1} u[k]
\]

\[
\Rightarrow y[k] = \sum_{n=0}^{\infty} g[n]u[k-n] \text{ with } g[0] = 0; \quad g[n] = (0.5)^{n-1}, \quad n \geq 1
\]

**Thus, we can convert difference forms to convolution forms with the operator form**

- In general for every difference form we write, the impulse response of the LTI system has a specific structure (mathematical form)
  - This is natural since difference forms possess a structure

**We can always convert a convolution form to the difference form by imposing a mathematical form on the impulse response** \( g[k] \)

z-Transform

Since the first step is to transform the time-domain signals to z-domain, we shall study how to carry out this transform.

- The one-sided z-transform of a signal \( f[k] \) (\( f[k] = 0, k < 0 \)) is defined as:
  \[
  F(z) = \sum_{k=0}^{\infty} f[k]z^{-k}
  \]

Observe that now the signal is characterized in terms of a variable ‘z’, which is a complex variable \( z = \sigma + j\omega \)

- To get back \( f[k] \), an inverse transform (a contour integral) may be used:
  \[
  f[k] = \frac{1}{2\pi j} \oint F(z)z^{-k-1} \, dz
  \]
  - In practice, we rarely follow this route to get back \( f[k] \). We rather use a simpler procedure, which we will explain later.

- We will now understand how some elementary signals appear in the z-domain

**Some examples**

- **Unit impulse:** \( f[k] = \delta[k] = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \) \( \Rightarrow F(z) = 1 \)

- **Unit step:** \( f[k] = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases} \)
  \[
  \Rightarrow F(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}}; \quad \text{ROC } |z^{-1}| < 1
  \]
  - The ROC stands for region of convergence

- **Exponential:** \( f[k] = a^k, \quad |a| < 1, \quad k \geq 0 \)
  \[
  \Rightarrow F(z) = \sum_{k=0}^{\infty} a^k z^{-k} = \frac{1}{1 - az^{-1}}; \quad \text{ROC } |az^{-1}| < 1
  \]

- **Sinusoid:** \( f[k] = \sin(\omega_0 k) \)
  \[
  \Rightarrow F(z) = \sum_{k=0}^{\infty} \sin(\omega_0 k)z^{-k} = \frac{z \sin(\omega_0)}{z^2 - 2z \cos(\omega_0) + 1}
  \]
Inverse z-Transform

- Quite often, we would be interested in arriving at \(x[k]\) given its z-transform
  - For this purpose, we usually break up the given \(X(z)\) into terms for which the inverse transform is easily known.
- Examples: (it is important to specify the time range for the final solution)

  **Example 1:**
  \[X(z) = \frac{1}{(1 - 0.5z^{-1})(1 - 0.3z^{-1})} = \frac{2.5}{1 - 0.5z^{-1}} - \frac{1.5}{1 - 0.3z^{-1}}\]
  \[\Rightarrow x[k] = 2.5(0.5)^k - 1.5(0.3)^k, \quad k \geq 0\] (a sum of exponentials)

  **Example 2:**
  \[X(z) = \frac{2z^{-1}}{(1 - z^{-1})^2}\]
  \[= 2z^{-2} \frac{z^{-1}}{(1 - z^{-1})^2}\]
  \[\Rightarrow x[k] = 2(k - 2), \quad k \geq 2\] (a delayed ramp)

Final and Initial Value Theorems

- One of the key advantages of analyzing signals in z-domain is that the signal (and the system) characteristics can be directly studied in the z-domain
  - Of particular interest are (i) signal’s growth/decay, (ii) the final value and (iii) initial value
  - The pole locations of \(X(z)\) give information on the growth and decay of the signal \(x[k]\).
  - The final value theorem is particularly useful in computing the gain of a stable system.

  **Final value theorem:**
  \[\lim_{k \to \infty} x[k] = \lim_{z \to 1} (1 - z^{-1})X(z)\]
  provided \((1 - z^{-1})X(z)\) has no poles on or outside the unit circle \(|z| = 1\)
  - The factor \((1 - z^{-1})\) takes care of the special signal, the step, which neither decays nor grows

  **Initial value theorem:**
  \[\lim_{k \to 0} x[k] = \lim_{z \to \infty} X(z)\] if it exists

Properties of z-Transform

- The z-transform has some useful properties which are useful for the analysis of LTI systems
  - **Linearity:** \(Z\{\alpha_1 x_1 + \alpha_2 x_2\} = \alpha_1 X_1(z) + \alpha_2 X_2(z)\) \(\forall \alpha_1, \alpha_2 \in \mathbb{C}\)
  - **Delay:**
    \[Z\{x[k - D]\} = z^{-D}X(z) + \sum_{k=-D}^{\infty} x[k]z^{-k}\] \(\forall D \geq 0\)
  - **Positive shift:**
    \[Z\{x[k + D]\} = z^{D}X(z) - \sum_{n=0}^{D-1} x[n]z^{-n}\] \(\forall D \geq 0\)
  - **Convolution:** (assuming causal sequences, but extensible to other cases)
    \[Z\left(\sum_{n=0}^{k} x[n]x[k-n]\right) = X_1(z)X_2(z)\]
    - The convolution property is one of the most useful properties in analysis of LTI systems
    - The input-output of an LTI system are related through convolution, which transforms to a product in the z-domain

Solving difference equations

- A useful application of z-transform is in solving the difference equations with constant coefficients and known initial conditions
  - Analogous to the use of Laplace Transforms in solving ODEs with constant coefficients
  - Only one-sided z-transforms are useful in solving difference equations

  **Example:** Solve \(x[k + 2] - 0.7x[k + 1] + 0.1x[k] = u[k], k \geq 0\)
  - The initial conditions are \(x[0] = 0, x[1] = 0\) and the forcing function \(u[k]\) is a unit step.

  **Solution:**
  \[z^2X(z) - z^2x[0] - zx[1] - 0.7(zX(z) -zx[0]) + 0.1X(z) = U(z)\]
  \[\Rightarrow X(z) = \frac{U(z)}{z^2 - 0.7z + 0.1}\]
  \[\Rightarrow x[k] = X(z)Z^{-1}\left(c_1 \frac{z}{z - 0.5} + c_2 \frac{z}{z - 0.2} + c_3 \frac{z}{z - 1}\right)\]
  \[= c_1(0.5)^k + c_2(0.2)^k + c_3, \quad k \geq 0\]
  where \(c_1 = -\frac{20}{3}, c_2 = \frac{25}{6}, c_3 = \frac{5}{2}\)
  The reader should verify that the solution satisfies the initial conditions

  Another approach is to expand \(X(z)\) as a power series in \(z^{-1}\) and compare the coefficients to get \(x[k]\)
Transfer Functions

- The most useful applications of z-transforms in linear systems theory is its ability to map convolution in time-domain to a product in z-domain
  - This is once again similar to the property of Laplace transforms for continuous-time signals
- The consequence is that the input-output relationship is an algebraic one
  \[ g[k] = \sum_{n=0}^{k} g[n]u[k-n] \]
  \[ Z\{g[k]\} = Z\left\{ \sum_{n=0}^{\infty} g[n]u[k-n] \right\} \implies Y(z) = G(z)U(z) \]
- The quantity \( G(z) \) is a function of the complex variable ‘z’ and denotes “how much” the input has been transferred to the output - hence acquires the name transfer function
- Note that \( G(z) = Z\{g[k]\} = \sum_{k=0}^{\infty} g[k]z^{-k} \)
  - A common definition of \( G(z) \) is that it is the ratio \( Y(z)/U(z) \) with zero initial conditions

Use of transfer functions

- The transfer function is useful in many ways
  - For a given input profile, one can compute \( Y(z) \) and then calculate the time-response using inverse z-transform.
  - Earlier, we saw how we could z-transform of a signal to know key characteristics of a signal. The same applies to the use of transfer functions. We can infer stability, oscillatory characteristics, damping, inverse response, etc. by merely examining \( G(z) \)
- The most important parameters of a transfer function are
  - **Poles**: Roots of denominator(\( G(z) \)) = 0 (a.k.a characteristic equation). The poles govern the stability of the system, shape/speed of response and the natural response of the system. They are solely a property of the system (in fact, its inertia).
  - **Zeros**: Roots of numerator(\( G(z) \)) = 0. The zeros tell us what class of inputs are completely blocked by the system. They also denote how the input interacts with the system
  - **Gain**: Defined as the change in output per unit change in input at steady-state. Determines the steady-state characteristic of the system. This is also known as the D.C. Gain. The gain

Poles

- We shall understand the influence of poles on a system's response below
- Examples:
  1. \( G(z) = K/(z-p_1) \). By definition \( g[n] = Z^{-1}\{G(z)\} = Kp_1^n, \ n \geq 0 \)
     - If \( |p| < 1 \), the IR decays at large times, implying stability
     - If \( |p| > 1 \), the IR blows up at large times, implying instability
     - If \( p_1 = 1 \), the IR remains constant (neither decays nor grows)
     - If \( p_1 = -1 \), the IR oscillates between +1 and -1 (remains bounded)
  2. Complex poles imply damped oscillatory impulse response
    - Purely imaginary poles (\( \zeta = 0 \)) produce purely oscillatory response
- Any t.f. can be broken up as sum of t.f.s with real and/or complex poles
  - Thus, the overall IR is the sum of the IR of the corresponding individual systems.
- For stability, the strict requirement is that all poles lie inside the unit circle
  - Systems with a single pole on the unit circle are considered to be marginally stable

Zeros

- The zeros denote the way the inputs interact with the system
- The effect of zero is to usually nullify or weaken the effect of a pole
- Zeros have no impact on the stability of the system!
Computing responses from transfer function

- We will learn by example how to compute responses from transfer functions.

**Example:** Given \( G(z) = \frac{z^{-1}}{1 - 0.8z^{-1} + 0.15z^{-2}} \) compute (i) impulse response and (ii) step response.

**Solution:**
- Impulse response:
  
  \[
  g[k] = Z^{-1} \{ G(z) \} = Z^{-1} \left\{ \frac{2.5z^{-1}}{1 - 0.5z^{-1} - 1.5z^{-1}} \right\} = 2.5(0.5)^k - 1.5(0.3)^{k-1}, \quad k \geq 1
  \]

- Step response:
  
  \[
  s[k] = Z^{-1} \left\{ G(z) \cdot \frac{1}{1 - z^{-1}} \right\} = \sum_{n=1}^{k} g[k-n]
  \]

From transfer function to FRF

- Earlier, we learnt that the FRF of a system is given by
  
  \[ G(\omega) = \sum_{n=0}^{\infty} g[n]e^{-j\omega n} \]
  
  which exists only if \( g[n] \) is absolutely convergent (i.e., for stable systems).

- From the expression it is clear that the FRF can be obtained by evaluating the TF on the unit circle \( z = e^{j\omega} \).

  \[
  G(\omega) = G(e^{j\omega}) = G(z)|_{z=e^{j\omega}}
  \]

  - The FRF tells us how much each frequency component in the input is scaled (in amplitude) and shifted (in time).

  - **Magnitude response** \( |G(e^{j\omega})| \): The magnitude of the FRF. It gives us the amplitude scaling at a given frequency.

  - **Phase response** \( \angle G(e^{j\omega}) \): The argument of the FRF. It gives us the phase shift at a given frequency.

State-space description

- Quite often, it happens that the measured output is not the same as the physical output (that we would like to measure).
  
  - For e.g., instead of measuring composition we measure temperature.

- In such cases, it is useful to write descriptions involving those “hidden” unmeasured quantities.

- The “hidden” (directly unobserved) quantities can be termed as **states** of a process since they internally characterize process conditions.

- The notion of states can be extended to all those variables which internally characterize a system.

  - Several such internal variables may exist, but we are only interested in a minimum subset.

  - These internal variables can be even fictitious (i.e., not physically meaningful) so long as they mathematically describe the system.

State-space models

- State-space models arise naturally in first-principles models.

- We now show an example of how a state-space model could be formed from a difference equation:
  
  \[
  y[k] + a_1 y[k-1] = b_1 u[k-1]
  \]

  - Assume we wished to measure \( x[k] \), but instead we measure \( y[k] \).

  - Assume further that these quantities are linearly related, i.e., \( y[k] = cx[k] \).

- Then, the state-space model is:
  
  \[
  x[k] + a_1 x[k-1] = b_1 u[k-1]
  \]

  which is conventionally re-written as,

  \[
  x[k+1] = -a_1 x[k] + b_1 u[k]
  \]

  \[
  y[k] = cx[k]
  \]

- Since the constant ‘c’ can be arbitrarily chosen (why?), the states can also be arbitrary! (alternatively state could be \( x[k]/\alpha \), then ‘c’ is \( \alpha c \)).

  - State-space descriptions are not unique as we will show formally.
Understanding the state-space model

- In general, the state-space model has the form
  
  \[
  \begin{align*}
  x[k + 1] &= A_dx[k] + B_du[k] \quad \text{(state equations)} \\
  y[k] &= Cx[k] + Du[k] \quad \text{(output equations)}
  \end{align*}
  \]

- The state equation describes the inertial part of the process
  - It tells us how the input affects the internal quantities of a system. Observe
    - Each state equation is a first-order difference equation
    - A one-sample delay exists between the state and the input

- The output equation quantifies
  - How the states affect measurements \(y[k]\) (thus, \(C\) could be thought of as a calibration matrix)
  - The direct effect of the input (bypassing the inertial effects) (thus, \(D\) represents feedthrough term)
  - The state-space model for a continuous-time system has a similar form, except that the state equations are derivatives (the output equation is still algebraic)

State-space models from T.F. models

- Traditionally state-space descriptions are introduced as the equivalent of breaking up an \(n\)th-order input-output difference equation into \(n\) first-order difference equations
  - Take, for example, \(y[k] + a_1 y[k-1] + a_2 y[k-2] = b_1 u[k-1]\)
    - Now introduce two fictitious quantities \(x_1[k] = y[k-1]; x_2[k] = y[k]\)
    - Then, we arrive at a state-space model
      \[
      \begin{bmatrix}
      x_1[k+1] \\
      x_2[k+1]
      \end{bmatrix} = \begin{bmatrix}
      0 & 1 \\
      -a_2 & -a_1
      \end{bmatrix} \begin{bmatrix}
      x_1[k] \\
      x_2[k]
      \end{bmatrix} + \begin{bmatrix}
      0 \\
      b_1
      \end{bmatrix} u[k]
      \]
      \[
      y[k] = \begin{bmatrix}
      0 & 1
      \end{bmatrix} x[k]
      \]
    - Naturally with a different choice of the states, we will obtain a different state-space model

- In general, any transformation \(x = Tw\), where \(T\) is a square non-singular matrix, will allow us to re-write the model in a new state-space

- To arrive at s.s. models from t.f. models, one first writes the difference equation and then casts it into a state-space form.